Algebraic geometry 1

Exercise Sheet 11

PD Dr. Maksim Zhykhovich Winter Semester 2025, 16.01.2026

Exercise 1. Let A be a commutative ring and let $I \subset A$ be an ideal. Show that the Krull dimensions of the quotient rings A/I and $A/\sqrt{I}$ are equal.
Hint: Recall from Commutative Algebra how one can describe the prime ideals of A/I in terms of prime ideals of A.

Exercise 2. (1) Let $S = \{a_1, \dots, a_r\}$ be a finite set of points in $\mathbb{P}^n$. Show that there exists a homogeneous polynomial $F \in K[X_0, \dots, X_n]$ of degree 1, such that $V(F) \cap S = \emptyset$.
(2) Let $X \subset \mathbb{P}^n$ be a projective algebraic set and let $X = X_1 \cup \cdots \cup X_r$, where X_i are irreducible components of X. Deduce from (1) that there exists a homogeneous polynomial $F \in K[X_0, \dots, X_n]$ of degree 1, such that the hyperplane $V(F)$ does not contain any of X_i, $i = 1, \dots, r$.

Recall that the degree of a projective algebraic set $X \subset \mathbb{P}^n$ is equal to $am!$, where $m = \dim X$ and a is a leading coefficient of the Hilbert polynomial of X.

Exercise 3. Denote by $\Sigma_{n,m}$ the image of the Segree embedding

$$\mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{(n+1)(m+1)-1}.$$

Determine the Hilbert polynomial and the degree of $\Sigma_{n,m}$.
Hint: Consider $\varphi : K[\{Z_{ij}\}_{0 \leq i \leq n, 0 \leq j \leq m}] \to K[X_0, \dots, X_n, Y_0, \dots Y_m]$, $Z_{ij} \mapsto X_i Y_j$, and show that $\operatorname{Ker} \varphi = I(\Sigma_{n,m})$.

Exercise 4. Determine the Hilbert polynomial and the degree of the image of the Veronese embedding

$$v_d : \mathbb{P}^n \to \mathbb{P}^N.$$